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ORIGINAL PAPER

An eigenfunction expansion of the non-selfadjoint Sturm–Liouville operator with a singular potential

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Abstract In this paper, we consider the operator L generated in $L^2(\mathbb{R}_+)$ by the differential expression

$$l(y) = -y'' + \left[\frac{\nu^2 - \frac{1}{4}}{x^2} + q(x)\right] y, \ \ x \in \mathbb{R}_+ := (0, \infty)$$

and the boundary condition

$$\lim_{x \to 0} x^{-\nu - \frac{1}{2}} y(x) = 1,$$

where q is a complex valued function and ν is a complex number with $Re\nu > 0$. We have proved a spectral expansion of L in terms of the principal functions under the condition

$$\underset{x \in \mathbb{R}_+}{Sup} \left\{ e^{\epsilon \sqrt{x}} |q(x)| \right\} < \infty, \ \ \epsilon > 0$$

taking into account the spectral singularities. We have also investigated the convergence of the spectral expansion.

Keywords Eigenvalues · Spectral singularities · Resolvent · Spectral expansion

Mathematics Subject Classification 47E05 · 34B05 · 34L05 · 47A10

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1 Introduction

The spectral analysis of a non-selfadjoint differential operators with continuous and discrete spectrum was investigated by Naimark [1] He showed the existence of spectral singularities in the continuous spectrum of the non-selfadjoint differential operator L_0, generated in $L_2(\mathbb{R}_+)$, by the differential expression

$$l_0(y) = -y'' + q(x)\,y,\ \ x \in \mathbb{R}_+ = [0,\infty) \tag{1.1}$$

with the boundary condition $y'(0) - hy(0) = 0$, where q is a complex valued function and $h \in C$. If the following condition

$$\int e^{\epsilon x}\,|q(x)|\,dx < \infty,\ \epsilon > 0$$

satisfies, then L_0 has a finite number of eigenvalues and spectral singularities with finite multiplicities.

Another approach for the discussion of the spectral analysis of L_0 was given by Marchenco [2]. Let E denote the set of all even entire functions of exponential type which are integrable over the real axis, and let E' denote the dual of E. We define

$$\varphi(f_i,\lambda) = \int_0^\infty f_i(x)\varphi(x,\lambda)dx, \quad i = 1,2$$

for any finite $f_1, f_2 \in L_2(R_+)$, where $\varphi(x,\lambda)$ is the solution of $l_0(y) = \lambda^2 y$, subject to the initial conditions $\varphi(0,\lambda) = 1,\ \varphi_x(0,\lambda) = h$. In [2] Marchenko showed that

$$\varphi(f_1,\lambda), \varphi(f_2,\lambda) \in E,$$

and there exists a functional $T \in E'$ such that

$$\int_0^\infty f_1(x) f_2(x) dx = T\left[\varphi(f_1,\lambda), \varphi(f_2,\lambda)\right] \tag{1.2}$$

T is the generalized spectral function of L_0. (1.2) is a generalization of the well-known Parseval equality for the singular selfadjoint differential operators, and it is called Marchenko-Parseval equality.

Lyance [3] has studied the effect of spectral singularities in the spectral expansion in terms of the principal functions for the operator L_0.

The Laurent expansion of the resolvents of the abstract non-selfadjoint operators in neighborhood of spectral singularities was investigated by Gasymov and Maksudov [4] and Maksudov and Allkhverdiev [5]. They also studied the effect of spectral singularities in the spectral analysis of these operators.

The spectral analysis of some classes of dissipative operators with spectral singularities was considered by Pavlov [6] using the theory of functional models [7] and scattering theory [8].

Some problems of spectral theory of differential and some others types of operators with spectral singularities were also studied in [9–12].

Discrete spectrum, principal functions and eigenfunction expansion of the quadratic pencil of Schrödinger operators were investigated in [13–15]. Spectral expansion of a non-selfadjoint differential operator on the whole axis was studied in [16].

Let us consider the operator L generated in $L_2\left(\mathbb{R}_+\right)$ by the differential expression

$$l\left(y\right)=-y''+\left[\frac{\nu^2-\frac{1}{4}}{x^2}+q\left(x\right)\right]y,\ \ x\in\mathbb{R}_+$$

and the boundary condition

$$\lim_{x\to 0}x^{-\nu-\frac{1}{2}}y\left(x\right)=1,$$

where q is a complex valued function and ν is a complex number with $Re\nu>0$. In [17] it has been proved that the operator L has of a finite number and spectral singularities, and each of them is of finite multiplicity under the condition

$$\underset{x\in\mathbb{R}_+}{Sup}\left\{e^{\epsilon\sqrt{x}}\left|q(x)\right|\right\}<\infty,\quad \epsilon>0 \tag{1.3}$$

Moreover, the properties of the principal functions corresponding to the eigenvalues and the spectral singularities of L have been obtained.

In this paper, which is a continuation of [17], we investigated the spectral functions, using a contour integral method and the regularization of divergent integrals, using summability factors.

2 Jost solution and jost function

Let us consider the equation

$$-y''+\left[\frac{\nu^2-\frac{1}{4}}{x^2}+q\left(x\right)\right]y=k^2y,\ \ x\in\mathbb{R}_+. \tag{2.1}$$

We have previously considered in [17] that q is almost everywhere continuous in $\mathbb{R}_+$ and satisfies the following [18, Chap.3]:

$$\int_a^\infty\left|q\left(x\right)\right|dx<\infty,\ \int_0^{a'}x\left|q\left(x\right)\right|dx<\infty,\ \ \left(a,a'>0\right). \tag{2.2}$$

Let $\varphi(x,k,\nu)$ and $f(x,k,\nu)$ denote the solutions of (2.1) satisfying the conditions

$$\lim_{x\to 0} x^{-\nu-\frac{1}{2}} y(x) = 1 \tag{2.3}$$

and

$$\lim_{x\to\infty} e^{-ikx} y(x) = 1 \tag{2.4}$$

respectively. The solution $f(x,k,\nu)$ is called Jost solution of (2.1). Note that, under the condition (2.2) the solution $\varphi(x,k,\nu)$ is an entire function of k and Jost solution is an analytic function of k in $\mathbb{C}_+ := \{k : k \in \mathbb{C}, \operatorname{Im} k > 0\}$ and continuous in $\overline{\mathbb{C}}_+ = \{k : k \in \mathbb{C}, \operatorname{Im} \lambda \geq 0\}$ [18, Chap. 4]. Moreover Jost solution also satisfies

$$f(x,k,\nu) = e^{ikx}[1 + o(1)], \;\; k \in \overline{\mathbb{C}}_+, \operatorname{Re}\nu > 0, \;\; x \to \infty. \tag{2.5}$$

Let us consider the function

$$f_0(x,k,\nu) = \sqrt{\frac{1}{2}\pi k x}\, e^{-\frac{1}{2}i\pi\left(\nu+\frac{1}{2}\right)} H_\nu^2(kx), \tag{2.6}$$

where $H_\nu^2(kx)$ is the Hankel function of second kind. It is obvious that the function $f_0(x,k,\nu)$ is the solution of the equation

$$-y'' + \frac{\nu^2 - \frac{1}{4}}{x^2} y = k^2 y.$$

Under the condition (2.2) Jost solution has the representation

$$f(x,k,\nu) = f_0(x,k,\nu) + \int_x^\infty K^{(\nu)}(x,t)\, f_0(t,k,\nu)\, dt, \tag{2.7}$$

where, the kernel $K^{(\nu)}(x,t)$ may be expressed in terms of q [19, Chap. 5] and satisfies

$$\left|K^{(\nu)}(x,t)\right| \leq c e^{-\frac{\epsilon}{2}(x+t)} \tag{2.8}$$

where $c > 0$.

We will denote the Wronskian of the solutions $f(x,k,\nu)$ and $\varphi(x,k,\nu)$ by $f_\nu(k)$ i.e.,

$$f_\nu(k) = W[f(x,k,\nu), \varphi(x,k,\nu)], \;\; k \in \overline{\mathbb{C}}_+, \operatorname{Re}\nu > 0. \tag{2.9}$$

The function f_ν is called Jost function of L. Under the condition (2.2) Jost function is analytic with respect to k in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}}_+$ and

$$f_\nu(k) = 1 + o(1), \quad k \in \overline{\mathbb{C}}_+, \ \mathrm{Re}\, \nu > 0, \ |k| \to \infty, \tag{2.10}$$

holds [18, Chap.5].

3 THE spectrum of L

By $\sigma_d(L)$ and $\sigma_{ss}(L)$ we denote the eigenvalues and spectral singularities of L, respectively. We have previously shown [17] that

$$\sigma_d(L) = \left\{\lambda : \lambda = k^2, \ k \in \mathbb{C}_+, \ f_\nu(k) = 0\right\}$$
$$\sigma_{ss}(L) = \left\{\lambda : \lambda = k^2, k \in \mathbb{R}^*, \ f_\nu(k) = 0\right\}$$

where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

We have also previously obtained that [17]: Let $G(x, t, k, \nu)$ be the Green function of L, i.e.,

$$G(x, t, k, \nu) = \begin{cases} \frac{\varphi(t,k,\nu) f(x,k,\nu)}{f_\nu(k)}, & 0 < t < x \\ \frac{\varphi(x,k,\nu) f(t,k,\nu)}{f_\nu(k)}, & x \le t < \infty. \end{cases} \tag{3.1}$$

Under the condition (1.3), we know that has a finite number of eigenvalues and spectral singularities, and each of them is finite multiplicity [17]. Let $\lambda_1 = k_1^2, \ldots, \lambda_\alpha = k_\alpha^2$ and $\lambda_{\alpha+1} = k_{\alpha+1}^2, \ldots, \lambda_n = k_n^2$ denote the eigenvalues and the spectral singularities of L with multiplicities $m_1, \ldots, m_\alpha$ and $m_{\alpha+1}, \ldots, m_n$ respectively.

We will also need the Hilbert spaces

$$H_m = \left\{ f : \int_0^\infty (1+x)^{2m} |f(x)|^2 \, dx < \infty \right\}, \ m = 0, 1, \ldots,$$
$$H_{-m} = \left\{ g : \int_0^\infty (1+x)^{-2m} |g(x)|^2 \, dx < \infty \right\}, \ m = 0, 1, \ldots,$$

with

$$\|f\|_m^2 = \int_0^\infty (1+x)^{2m} |f(x)|^2 \, dx, \quad \|g\|_m^2 = \int_0^\infty (1+x)^{-2m} |g(x)|^2 \, dx,$$

respectively. It is obvious that $H_0 = L^2(\mathbb{R}_+)$ and

$$H_{m+1} \subsetneqq H_m \subsetneqq L^2(\mathbb{R}_+) \subsetneqq H_{-m} \subsetneqq H_{-(m+1)}, \quad m = 1, 2, \ldots,$$

and H_{-m} is isomorphic to the dual of $H_m : H_m^{'} \sim H_{-m}$.

We have previously shown that [17]:

$$\Phi_j\left(., k_p, \nu\right) \in L^2\left(\mathbb{R}_+\right), \quad j = 0, 1, \ldots, m_p - 1, \quad p = 1, \ldots, \alpha, \tag{3.2}$$

$$\Phi_j\left(., k_p, \nu\right) \in H_{-(j+1)}, \quad j = 0, 1, \ldots, m_p - 1, \quad p = \alpha + 1, \ldots, n \tag{3.3}$$

where

$$\Phi_j\left(., k_p, \nu\right) = \sum_{\beta=0}^{j} A_{j-\beta}(k_p)\frac{1}{\beta!}\left\{\frac{\partial^\beta}{\partial k^\beta} f\left(x, k, \nu\right)\right\}_{k=k_p}$$
$$j = 0, 1, \ldots, m_p - 1, \quad p = 1, \ldots, \alpha, \alpha + 1, \ldots, n \tag{3.4}$$

The functions $\Phi_j\left(., k_p, \nu\right), \ j = 0, 1, \ldots, m_p - 1, \quad p = 1, \ldots, \alpha$ and $\Phi_j\left(., k_p, \nu\right)$, $j = 0, 1, \ldots, m_p - 1, \quad p = \alpha + 1, \ldots, n$ are the principal functions corresponding to the eigenvalues and the spectral singularities of L, respectively.

4 THE spectral expansion

Let $C_0^\infty\left(\mathbb{R}_+\right)$ denote the set of infinetely differentiable functions in $\mathbb{R}_+$ with compact support. Then

$$\psi(x) = R(L)R^{-1}(L)\psi(x) = R(L)(L - k^2 I)\psi(x)$$
$$\psi(x) = \int_0^\infty G(x, t, k, \upsilon)\left[-\psi^{''}(t) + \frac{\nu^2 - \frac{1}{4}}{t^2}\psi(t) + q(t)\psi(t) - k^2\psi(t)\right]dt$$

for each $\psi \in C_0^\infty\left(\mathbb{R}_+\right)$. Thus we obtain

$$\frac{\psi(x)}{k} = \frac{1}{k}\int_0^\infty G(x, t, k, \upsilon)\theta(t)dt$$
$$+\frac{1}{k}\int_0^\infty G(x, t, k, \upsilon)\left(\frac{\nu^2 - \frac{1}{4}}{t^2}\right)\psi(t)dt - kD(x, k, \upsilon) \tag{4.1}$$

where

$$\theta(t) = -\psi^{''}(t) + q(t)\psi(t), \quad D(x, k, \upsilon) = \int_0^\infty G(x, t, k, \upsilon)\psi(t)dt$$

Let γ_r denote the disc with center at the origin having radius r; let $\partial\gamma_r$ be the boundary of γ_r. r will be chosen so that all eigenvalues and spectral singularities of L are in γ_r. $P_{r\eta}$ denotes the part of γ_r lying in the strip $|\text{Im}\,\lambda| \leq \eta$ and $\gamma_{r\eta} = \gamma_{r\eta}^+ \cup \gamma_{r\eta}^-$, where

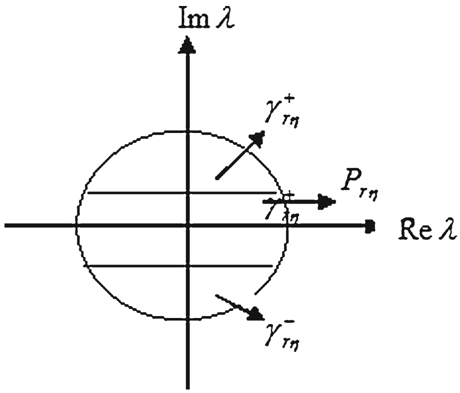

Fig. 1 Eigenvalues and spectral singularities are on the semicircles only

$\gamma_{r\eta}^{+}$ and $\gamma_{r\eta}^{-}$ are the parts of $\gamma_r \backslash P_{r\eta}$ in the upper and the lower half-planes, respectively (see Fig. 1). We mention that $\gamma_{r\eta}^{-}$ has not any eigenvalues and spectral singularities.

We chose η so small that $P_{r\eta}$ does not contain any eigenvalues of L.

So we easily see that

$$\partial\gamma_r = \partial\gamma_{r\eta} \cup \partial P_{r\eta} \tag{4.2}$$

From (4.1) we get

$$\begin{aligned}\psi(x) = &\frac{1}{2\pi i}\int\limits_{\partial\gamma_r}\left\{\frac{1}{k}\int\limits_0^\infty G(x,t,k,\upsilon)\theta(t)dt\right\}dk\\ &+\frac{1}{2\pi i}\int\limits_{\partial\gamma_r}\left\{\frac{1}{k}\int\limits_0^\infty G(x,t,k,\upsilon)\left(\frac{\nu^2-\frac{1}{4}}{t^2}\right)\psi(t)dt\right\}dk\\ &-\frac{1}{2\pi i}\int\limits_{\partial\gamma_r} kD(x,k,\upsilon)dk\end{aligned} \tag{4.3}$$

Using (2.10), (3.1) and Jordan's lemma, we see that the first term of the right hand side of (4.3) vanishes as $r \to \infty$. The same result holds for the second term. This can be obtained from (2.10) utilizing integration by parts. Then considering (4.2) we obtain

$$\psi(x) = -\lim_{\substack{r\to\infty\\ \eta\to 0}}\frac{1}{2\pi i}\int\limits_{\partial\gamma_{r\eta}} kD(x,k,\upsilon)dk - \lim_{\substack{r\to\infty\\ \eta\to 0}}\frac{1}{2\pi i}\int\limits_{\partial P_{r\eta}} kD(x,k,\upsilon)dk \tag{4.4}$$

We easily obtain that the first integral in (4.4) gives

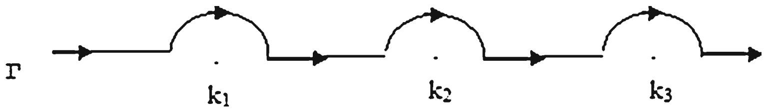

Fig. 2 Isolated real zeros

$$\lim_{\substack{r\to\infty\\ \eta\to 0}} \frac{1}{2\pi i}\int_{\partial\gamma_{r\eta}} kD(x,k,\upsilon)dk = \sum_{i=1}^{\alpha} \operatorname*{Re\,s}_{\lambda=\lambda_i}[kD(x,k,\nu)]$$

where

$$D(x,k,\nu)=\int_0^{\infty} G(x,t,k,\upsilon)\psi(t)dt.$$

Let Γ be the contour which is isolates the real zeros of f by semicircles with centers at $k_i,\ i=1,2,\ldots,\alpha$ having the same radius δ_0 in the upper-half plane. The radius of semicircles being chosen so small that their diameters are mutually disjoint and do not contain the point $\lambda=0$ (see Fig. 2).

As it is easily seen from Fig. 1, we find

$$\lim_{\substack{r\to\infty\\ \eta\to 0}} \frac{1}{2\pi i}\int_{\partial P_{r\eta}} kD(x,k,\upsilon)dk = \frac{1}{2\pi i}\int_{\Gamma} kD(x,k,\upsilon)dk$$

Therefore (4.4) can be written as

$$\psi(x) = -\sum_{i=1}^{\alpha} \operatorname*{Re\,s}_{\lambda=\lambda_i}[kD(x,k,\nu)] - \frac{1}{2\pi i}\int_{\Gamma} kD(x,k,\upsilon)dk \tag{4.5}$$

Theorem 4.1 *For every* $\psi \in C_0^{\infty}(\mathbb{R}_+)$

$$\psi(x) = \sum_{i=1}^{\alpha}\left\{\left(\frac{\partial}{\partial k}\right)^{m_i-1}[a_i(k)\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)]\right\}_{k=k_i} + \frac{1}{2\pi i}\int_{\Gamma}\frac{kf(0,k,\upsilon)}{f_{\upsilon}(k)}\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)\,dk \tag{4.6}$$

$$0 = \sum_{i=1}^{\alpha}\left\{\left(\frac{\partial}{\partial k}\right)^{m_i-1}[b_i(k)\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)]\right\}_{k=k_i} + \frac{1}{2\pi i}\int_{\Gamma}\frac{f(0,k,\upsilon)}{f_{\upsilon}(k)}\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)\,dk \tag{4.7}$$

where

$$a_i(k) = -\frac{k(k-k_i)^{m_i} f(0,k,\upsilon)}{(m_i-1)!\, f_\upsilon(k)},\ i = 1,\ldots,j \tag{4.8}$$

$$b_i(k) = -\frac{(k-k_i)^{m_i} f(0,k,\upsilon)}{(m_i-1)!\, f_\upsilon(k)},\ i = 1,\ldots,k \tag{4.9}$$

and

$$\Phi(\psi,k,\nu) = \int_0^\infty \psi(t)\Phi(x,k,\upsilon)dt$$

Proof Let $B(x,k,\nu)$ be the solution of (2.1) subject to the initial conditions $\lim_{x\to 0} x^{-\upsilon-\frac{1}{2}} y(x) = 1,\ \lim_{x\to\infty} e^{-ikx} y(x) = 1$ Then

$$G(x,t,k,\upsilon) = \frac{f(0,k,\upsilon)}{f_\upsilon(k)} \Phi(x,k,\nu)\,\Phi(t,k,\nu) + a(x,t,k;\nu) \tag{4.10}$$

where

$$a(x,t,k,\nu) = \begin{cases} B(x,k,\nu)\Phi(t,k,\nu), & 0 < t < x \\ B(t,k,\nu)\Phi(x,k,\nu), & x \le t < \infty \end{cases}$$

and $a(x,t,k,\nu)$ is an entire function of k. From (4.5) and (4.10) we obtain (4.6). Writing (4.1) as

$$\frac{\psi(x)}{k^2} = \frac{1}{k^2}\int_0^\infty G(x,t,k,\upsilon)\theta(t)dt + \frac{1}{k^2}\int_0^\infty G(x,t,k,\upsilon) \times \left(\frac{\nu^2-\frac{1}{4}}{t^2}\right)\psi(t)dt - D(x,k,\upsilon)$$

and repeating the calculation as we done for (4.1), we have (4.7). □

Since the contour Γ in (4.6) and (4.7) do not coincide with the continuous spectrum of L, these formulae contains non-spectral objects. The purpose of this article is to transform (4.6) and (4.7) a into two-fold spectral expansion with respect to the principal functions of L.

Theorem 4.2 *For any* $\psi \in C_0^\infty(\mathbb{R}_+)$ *there exists a constant* $c > 0$ *so that*

$$\int_{-\infty}^\infty |k\Phi(\psi,k,\nu)|^2\, dk \le c\int_0^\infty |\psi(x)|^2\, dx \tag{4.11}$$

Proof From (3.4) we get

$$\Phi_j\left(\psi, k_p, \nu\right) = \sum_{\beta=0}^{j} A_{j-\beta}(k_p)\frac{1}{\beta!}\left\{\frac{\partial^\beta}{\partial k^\beta} f\left(\psi, k, \nu\right)\right\}_{k=k_p} \tag{4.12}$$

where

$$f\left(\psi, k, \nu\right) = \int_0^\infty \psi(x) f\left(x, k, \nu\right) dx.$$

Using (2.7), we obtain

$$\begin{aligned} f\left(\psi, k, \nu\right) &= \int_0^\infty \left\{ f_0\left(x, k, \nu\right) + \int_x^\infty K^\upsilon(x,t) f_0\left(t, k, \nu\right) dt \right\} \psi(x) dx \\ &= \int_0^\infty \psi(x) f_0\left(x, k, \nu\right) dx + \int_0^\infty \int_x^\infty \psi(x) K^\upsilon(x,t) f_0\left(x, k, \nu\right) dt dx. \end{aligned}$$

Changing the order of integration, we get

$$f\left(\psi, k, \nu\right) = \int_0^\infty \left\{(I + K^\upsilon)\psi(t)\right\} f_0\left(t, k, \nu\right) dt \tag{4.13}$$

in which the operator I is the unit operator, and K^υ is the operator defined by

$$K^\upsilon \psi(t) = \int_0^t K^\upsilon(x,t)\psi(x) dx.$$

From (2.8) we understand K^υ is a compact operator in $L^2\left(\mathbb{R}_+\right)$. Thus $(I + K^\upsilon)$ is a continuous and one-to-one on $L^2\left(\mathbb{R}_+\right)$. Using the Parseval's equality for the Fourier transforms and (4.13) we get

$$\int_{-\infty}^\infty \left|f(\psi, k, \upsilon)\right|^2 dk \le c_1 \int_0^\infty \left|\psi(x)\right|^2 dx \tag{4.14}$$

where $c_1 > 0$ is a constant.

The proof of the lemma is completed by (2.10) and (4.14). □

By the preceding lemma for every function $\psi \in L^2(\mathbb{R}_+)$ the limit

$$\Phi(\psi, k, \nu) = \lim_{N\to\infty} \int_0^N \psi(x)\Phi(x, k, \nu)\, dx$$

exists in the sense of convergence in the mean square, relative to the measure k^2dk on the real axis; that is,

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \left| \Phi(\psi, k, \nu) - \int_0^N \psi(x)\Phi(x, k, \nu)\, dx \right|^2 k^2 dk = 0 \tag{4.15}$$

Since $C_0^\infty(\mathbb{R}_+)$ is dense in $L^2(\mathbb{R}_+)$, the estimate (4.11) may be extended onto $L^2(\mathbb{R}_+)$ for any $\psi \in L^2(\mathbb{R}_+)$ as

$$\int_{-\infty}^{\infty} |k\Phi(\psi, k, \upsilon)|^2\, dk \leq c \int_0^{\infty} |\psi(x)|^2\, dx \tag{4.16}$$

where $\Phi(\psi, k, \upsilon)$ must be understood in the sense of (4.15). We shall need a generalization of this estimate.

Theorem 4.3 *If $\psi \in H_m$, then $\Phi(\psi, k, \upsilon)$ has a derivative of order (m-1) which is absolutely continuous of every finite subinterval of the real axis and satisfies*

$$\int_{-\infty}^{\infty} \left| \left(\frac{d}{dk}\right)^n [k\Phi(\psi, k, \upsilon)]\, dk \right|^2 \leq c_n \int_0^{\infty} (1+x)^{2n} |\psi(x)|^2\, dx \tag{4.17}$$

where $c_n > 0$ are constants, $n = 1, \ldots, m$.

The proof is similar to that of Theorem 4.2.

In order to transform (4.6) and (4.7) into the spectral expansion of L, we have to reform the integrals over Γ onto the real axis.

Since the spectral singularities of L are the zeros of f, the integrals over the real axis are divergent in the norm of $L^2(\mathbb{R}_+)$. Now we will investigate the convergence of these integrals in a norm which is weaker than the norm of $L^2(\mathbb{R}_+)$. For this purpose we will use the technique of regularization of divergent integrals. So we define the following summability factor:

$$F_{p\beta}(k) = \begin{cases} \frac{(k-k_p)^\beta}{\beta!}, & \left|k - k_p\right| < \delta,\ p = \alpha+1, \ldots, n \\ 0, & \left|k - k_p\right| \geqslant \delta,\ p = \alpha+1, \ldots, n \end{cases} \tag{4.18}$$

with $\delta > \delta_0$. We can choose $\delta > 0$ so small that the $\delta-$neighborhoods of k_p, $p = \alpha + 1, \ldots, n$ have no common points and do not contain the point $k = 0$. Define the functional

$$F\{g(k)\} = g(k) - \sum_{p=\alpha+1}^{n} \sum_{\beta=0}^{m_p-1} \left\{ \left(\frac{d}{dk} \right)^{\beta} g(k) \right\}_{k=k_p} F_{p\beta}(k) \tag{4.19}$$

where g is chosen so that the right hand side of the above formulae is meaningful. It is evident from (4.18) that $k_{\alpha+1}, \ldots, k_n$ are the roots of $F\{g(k)\} = 0$ at least of orders $m_{\alpha+1}, \ldots, m_n$.

In the following we will use the operators

$$P\psi(x) = \frac{1}{2\pi i} \int_{\Gamma} \frac{kf(0,k,\upsilon)}{f_{\upsilon}(k)} \Phi(x,k,\nu)\, \Phi(\psi,k,\nu)\, dk \tag{4.20}$$

and

$$\begin{aligned} I\psi(x) = {} & \frac{1}{2\pi i} \sum_{p=\alpha+1}^{n} \sum_{\beta=0}^{m_p-1} \left\{ \left(\frac{\partial}{\partial k} \right)^{\beta} \left[\Phi(x,k,\nu)\, \Phi(\psi,k,\nu) \right] \right\}_{k=k_p} \\ & \times \int_{\Gamma} \frac{kf(0,k,\upsilon)}{f_{\upsilon}(k)} F_{p\beta}(k) dk \\ & + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{kf(0,k,\upsilon)}{f_{\upsilon}(k)} F\{\Phi(x,k,\nu)\, \Phi(\psi,k,\nu)\}\, dk \end{aligned}$$

Since under the condition (1.3) $f(0,k,\upsilon)$ has an analytic continuation to the half-planes $\operatorname{Im} k > -\frac{\epsilon}{2}$, we get

$$P\psi = I\psi$$

for $\psi \in C_0^{\infty}(\mathbb{R}_+)$.

Theorem 4.4 *For each* $\psi \in H_{(m_0+1)}$, *there exist a constant* $c > 0$ *such that*

$$\|I\psi\|_{-(m_0+1)} \leq c_1 \|\psi\|_{(m_0+1)} \tag{4.21}$$

where $m_0 = \max\{m_{\alpha+1}, \ldots, .m_n\}$.

Proof Define

$$\Lambda_p = (k_p - \delta, k_p + \delta), \quad p = \alpha+1, \ldots, n \tag{4.22}$$

Then $0 \notin \Lambda_p,\ p = \alpha+1, \ldots, n$. Using the integral form of remainder in the Taylor formula, we get

$$F\left\{\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)\right\} = \begin{cases} \Phi(x,k,\nu)\Phi(\psi,k,\nu)\,,\ k\in\Lambda_0 \\ \frac{1}{(m_p-1)!}\int\limits_{k_p}^{k}(k-\xi)^{m_p-1}\left\{\left(\frac{\partial}{\partial\xi}\right)^{m_p}\left[\Phi(x,k,\xi)\,\Phi(\psi,k,\xi)\right]\right\}d\xi\ ,\ k\in\Lambda_p \end{cases} \tag{4.23}$$

where $\Lambda_0 = R\backslash\left\{\bigcup_{p=\alpha+1}^{n}\Lambda_p\right\}$.

If we use the notation

$$I_p\psi(x) = \frac{1}{2\pi i}\int\limits_{\Lambda_p}\frac{kf(0,k,\upsilon)}{f_\upsilon(k)}F\left\{\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)\right\}dk \quad p=\alpha+1,\ldots,n$$

$$\widetilde{I}\psi(x) = \frac{1}{2\pi i}\sum_{p=\alpha+1}^{n}\sum_{\beta=0}^{m_p-1}\left\{\left(\frac{\partial}{\partial k}\right)^{\beta}\left[\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)\right]\right\}_{k=k_p} \times\int\limits_{\Gamma}\frac{kf(0,k,\upsilon)}{f_\upsilon(k)}F_{p\beta}(k)dk$$

we obtain

$$I = I_{\alpha+1}+\cdots+I_n+\widetilde{I} \tag{4.24}$$

from (4.22) and (4.23). We now show that each of the operators $I_{\alpha+1},\ldots,I_n$ and $\widetilde{I}$ is continuous from $H_{(m_0+1)}$ into $H_{-(m_0+1)}$. We start from with $\widetilde{I}$. From (4.18) we obtain the absolute convergence of

$$\int\limits_{\Gamma}\frac{kf(0,k,\upsilon)}{f_\upsilon(k)}F_{p\beta}(k)dk.$$

Using (3.3) and the isomorphism $H_{-m_0}\sim H'_{m_0}$ we see that $\widetilde{I}$ is continuous from H_{m_0} into H_{-m_0} or from $H_{(m_0+1)}$ into $H_{-(m_0+1)}$. Hence there exists a constant $\widetilde{c}>0$ such that

$$\left\|\widetilde{I}\psi(x)\right\|_{-(m_0+1)}\le\widetilde{c}\,\|\psi\|_{(m_0+1)} \tag{4.25}$$

for any $\psi\in H_{(m_0+1)}$.

Next we want to show the continuity of $I_p,\ p=\alpha+1,\ldots,n$ from $H_{(m_0+1)}$ into $H_{-(m_0+1)}$. From (4.24) we see that

$$I_p\psi(x) = \frac{1}{2\pi i(m_p-1)!}\int\limits_{\Lambda_p}\frac{kf(0,k,\upsilon)}{f_\upsilon(k)}\int\limits_{k_p}^{k}(k-\xi)^{m_p-1} \times\left\{\left(\frac{\partial}{\partial\xi}\right)^{m_p}\left[\Phi(x,k,\upsilon)\,\Phi(\psi,k,\upsilon)\right]\right\}d\xi dk \tag{4.26}$$

Interchanging the order of integration, we get

$$\begin{aligned} I_p\psi(x) = \frac{1}{2\pi i(m_p-1)!}&\left\{\int\limits_{k_p}^{k_p+\delta}\int\limits_{\xi}^{k_p+\delta}\left\{\left(\frac{\partial}{\partial\xi}\right)^{m_p}\left[\Phi(x,\xi,\upsilon)\,\Phi(\psi,\xi,\upsilon)\right]\right\}\right. \\ &\times(k-\xi)^{m_p-1}\frac{kf(0,k,\upsilon)}{f_\upsilon(k)}dkd\xi \\ &-\int\limits_{k_p-\delta}^{k_p}\int\limits_{k_p-\delta}^{\xi}\left\{\left(\frac{\partial}{\partial\xi}\right)^{m_p}\left[\Phi(x,\xi,\upsilon)\,\Phi(\psi,\xi,\upsilon)\right]\right\} \\ &\times(k-\xi)^{m_p-1}\frac{kf(0,k,\upsilon)}{f_\upsilon(k)}dkd\xi. \end{aligned}$$

Since k_p is a zero of $f_\upsilon(k)$ order m_p, there exists a continuous function f_p such that $f_p(k_p)\neq 0$ and $f(k)=(k-k_p)^{m_p}f_p(k)$. On the other hand,

$$\left|\int\limits_{\xi}^{k_p+\delta}(k-\xi)^{m_p-1}\frac{kf(0,k,\upsilon)}{f_\upsilon(k)}dk\right| \le h_p^{(1)}(\xi)\left[\ln\delta-\ln(\xi-k_p)\right] \tag{4.27}$$

if $\xi > k_p$, and

$$\left|\int\limits_{k_p-\delta}^{\xi}(k-\xi)^{m_p-1}\frac{kf(0,k,\upsilon)}{f_\upsilon(k)}dk\right| \le h_p^{(2)}(\xi)\left[\ln(k_p-\xi)-\ln\delta\right] \tag{4.28}$$

if $\xi < k_p$, where

$$h_p^{(1)}(\xi) = \max_{k\in[\xi,k_p+\delta]}\left|\frac{kf(0,k,\upsilon)}{f_\upsilon(k)}\right|, \qquad h_p^{(2)}(\xi) = \max_{k\in[k_p-\delta,\xi]}\left|\frac{kf(0,k,\upsilon)}{f_\upsilon(k)}\right|.$$

(4.27) and (4.28) show that $I_p,\ p=\alpha+1,\ldots,n$ are integral operators with kernels having logarithmic singularities, i.e., weak singularities.

(4.26) can be written as

$$I_p\psi(x) = \int\limits_{\Lambda_p}\sum_{s=0}^{m_p}b_{sp}(x,\xi)\left\{\left(\frac{d}{d\xi}\right)^{s}\Phi(\psi,\xi,\upsilon)\right\}d\xi.$$

Define

$$B_{sp} = \int_0^\infty \int_{\Lambda_p} \left| \frac{b_{sp}(x,\xi)}{(1+x)^{m_0+1}} \right|^2 d\xi dx.$$

We see that $B_{sp} < \infty$, by (3.3), (4.27) and (4.28). Since

$$\begin{aligned} \left\| I_p \psi \right\|^2_{-(m_0+1)} &= \int_0^\infty \left| \frac{I_p \psi(x)}{(1+x)^{m_0+1}} \right|^2 dx \\ &\leq \sum_{s=0}^{m_p} \int_0^\infty \int_{\Lambda_p} \frac{b_{sp}(x,\xi)}{(1+x)^{m_0+1}} d\xi dx \int_{\Lambda_p} \left(\frac{d}{d\xi} \right)^s \Phi(\psi, \xi, \upsilon)\, d\xi \\ &= \sum_{k=0}^{m_p} B_{sp} \int_{\Lambda_p} \left| \left(\frac{d}{d\xi} \right)^s \Phi(\psi, \xi, \upsilon) \right|^2 d\xi \end{aligned}$$

Utilizing (4.16) and (4.17) we obtain

$$\left\| I_p \psi \right\|_{-(m_0+1)} \leq c_p \left\| \psi \right\|_{m_0} \leq c_p \left\| \psi \right\|_{(m_0+1)}, \quad p = \alpha+1, \ldots, n \tag{4.29}$$

where c_p are constants.

Lastly we consider the operator I_0 which is defined by

$$I_0 \psi = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \varkappa_0(k) \frac{k f(0,k,\upsilon)}{f_\upsilon(k)} \Phi(x,k,\upsilon)\, \Phi(\psi,k,\upsilon)\, dk \tag{4.30}$$

where $\varkappa_0$ is the characteristic function of the interval Λ_0. From (4.30), similar to the proof of Theorem 4.2, we get

$$\int_0^\infty \left| I_p \psi(x) \right|^2 dx \leq c_0 \int_0^\infty |\psi(x)|^2 dx,$$

where $c_0 > 0$ is a constant. Since

$$H_{(m_0+1)} \subsetneqq L^2(\mathbb{R}_+) \subsetneqq H_{-(m_0+1)},$$

we find

$$\left\| I_0 \psi \right\|_{-(m_0+1)} \leq \ c_0 \left\| \psi \right\|_{(m_0+1)} \tag{4.31}$$

From (4.24), (4.25), (4.29) and (4.31) we have

$$\|I\psi\|_{-(m_0+1)} \le c\,\|\psi\|_{(m_0+1)}$$

□

Then for every $\psi \in H_{(m_0+1)}$,

$$I\psi(x) = \frac{1}{2\pi i}\sum_{p=\alpha+1}^{n}\sum_{\beta=0}^{m_p-1}\left\{\left(\frac{\partial}{\partial k}\right)^{\beta}\left[\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)\right]\right\}_{k=k_p}$$
$$\times\int_{\Gamma}\frac{kf(0,k,\upsilon)}{f_{\upsilon}(k)}F_{p\beta}(k)dk$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{kf(0,k,\upsilon)}{f_{\upsilon}(k)}F\left\{\Phi(x,k,\upsilon)\,\Phi(\psi,k,\upsilon)\right\}dk \tag{4.32}$$

Let $a_p(k)$ denote any function which is defined and differentiable in a neighbourhood of k_p, and which satisfies the condition

$$\left\{\left(\frac{d}{dk}\right)^{m_p-1-\beta}a_p(k)\right\}_{k=k_p} = \frac{1}{2\pi i}\binom{m_p-1}{\beta}\int_{\Gamma}\frac{kf(0,k,\upsilon)}{f_{\upsilon}(k)}F_{p\beta}(k)dk$$
$$p=\alpha+1,\ldots,n \tag{4.33}$$

Then (4.32) can be written as

$$I\psi(x) = \sum_{p=\alpha+1}^{n}\left\{\left(\frac{\partial}{\partial k}\right)^{m_p-1}\left[a_p(k)\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)\right]\right\}_{k=k_p}$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{kf(0,k,\upsilon)}{f_{\upsilon}(k)}F\left\{\Phi(x,k,\upsilon)\,\Phi(\psi,k,\upsilon)\right\}dk \tag{4.34}$$

we shall also use the following integral operator [see (4.7)]:

$$Q\psi(x) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(0,k,\upsilon)}{f_{\upsilon}(k)}\Phi(x,k,\upsilon)\,\Phi(\psi,k,\upsilon)\,dk$$

$$J\psi(x) = \frac{1}{2\pi i}\sum_{p=\alpha+1}^{n}\sum_{\beta=0}^{m_p-1}\left\{\left(\frac{\partial}{\partial k}\right)^{\beta}\left[\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)\right]\right\}_{k=k_p}$$
$$\times\int_{\Gamma}\frac{f(0,k,\upsilon)}{f_{\upsilon}(k)}F_{p\beta}(k)dk$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(0,k,\upsilon)}{f_{\upsilon}(k)}F\left\{\Phi(x,k,\upsilon)\,\Phi(\psi,k,\upsilon)\right\}dk \tag{4.35}$$

It is evident that

$$Q\psi = J\psi,$$

for $\psi \in C_0^\infty(R_+)$.

Similar to Theorem 4.4, we find.

Theorem 4.5 *For every each $\psi \in H_{(m_0+1)}$, there exist a constant $c > 0$ such that*

$$\|J\psi\|_{-(m_0+1)} \le c\,\|\psi\|_{(m_0+1)}\,.$$

It is evident that, for every $\psi \in H_{(m_0+1)}$

$$J\psi(x) = \sum_{p=\alpha+1}^{n}\left\{\left(\frac{\partial}{\partial k}\right)^{m_p-1}\left[b_p(k)\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)\right]\right\}_{k=k_p} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(0,k,\upsilon)}{f_\upsilon(k)}F\left\{\Phi(x,k,\upsilon)\,\Phi(\psi,k,\upsilon)\right\}dk \tag{4.36}$$

where

$$\left\{\left(\frac{d}{dk}\right)^{m_p-1-\beta} b_p(k)\right\}_{k=k_p} = \frac{1}{2\pi i}\binom{m_p-1}{\beta}\int_{\Gamma}\frac{kf(0,k,\upsilon)}{f_\upsilon(k)}F_{p\beta}(k)dk$$
$$p=\alpha+1,\ldots,n \tag{4.37}$$

Theorem 4.6 *Under the condition (1.3) the following two-fold spectral expansion in terms of the principal functions of L holds,*

$$\psi(x) = \sum_{i=1}^{\alpha}\left\{\left(\frac{\partial}{\partial k}\right)^{m_i-1}\left[a_i(k)\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)\right]\right\}_{k=k_i} + \sum_{p=\alpha+1}^{n}\left\{\left(\frac{\partial}{\partial k}\right)^{m_p-1}\left[a_p(k)\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)\right]\right\}_{k=k_p} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{kf(0,k,\upsilon)}{f_\upsilon(k)}F\left\{\Phi(x,k,\upsilon)\,\Phi(\psi,k,\upsilon)\right\}dk \tag{4.38}$$

$$0 = \sum_{i=1}^{\alpha}\left\{\left(\frac{\partial}{\partial k}\right)^{m_i-1}\left[b_i(k)\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)\right]\right\}_{k=k_i} + \sum_{p=\alpha+1}^{n}\left\{\left(\frac{\partial}{\partial k}\right)^{m_p-1}\left[b_p(k)\Phi(x,k,\nu)\,\Phi(\psi,k,\nu)\right]\right\}_{k=k_p} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(0,k,\upsilon)}{f_\upsilon(k)}F\left\{\Phi(x,k,\upsilon)\,\Phi(\psi,k,\upsilon)\right\}dk \tag{4.39}$$

for every function $\psi \in H_{(m_0+1)}$. *The integrals in* (4.38) *and* (4.39) *converge in the norm of* $H_{-(m_0+1)}$ *where* a_i, b_i, F, a_p *and* b_p *defined by* (4.8), (4.9), (4.19), (4.33), *and* (4.37) *respectively.*

Proof We obtain (4.38) and (4.39) for $\psi \in C_0^{\infty}(R_+) \subset H_{(m_0+1)}$, by use of (4.6), (4.7), (4.20) and (4.34)–(4.36). The convergence of the integrals appearing in (4.38) and (4.39) in the norm of $H_{-(m_0+1)}$, has been given in Theorem 4.4 and Theorem 4.5. Since $C_0^{\infty}(R_+)$ is dense in $H_{(m_0+1)}$, the proof is completed. □

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